Introduction

Let X, Y be two real or complex Banach spaces. By $X = Y$ we mean that X and Y have the same elements and equivalent norms. By $Y \subset X$ we mean that Y is continuously embedded in X. As usual, we denote by $\mathcal{L}(X, Y)$ (in short, $\mathcal{L}(X)$ if $X = Y$) the space of all linear bounded operators from X to Y, endowed by the norm $\|T\|_{\mathcal{L}(X,Y)} := \sup\{\|Tx\|_Y : \|x\|_X = 1\}$.

The couple of Banach spaces (X, Y) is said to be an *interpolation couple* if both X and Y are continuously embedded in a Hausdorff topological vector space $\mathcal{V}$. In this case the intersection $X \cap Y$ is a linear subspace of $\mathcal{V}$, and it is a Banach space under the norm

$$\|v\|_{X\cap Y} := \max\{\|v\|_X,\ \|v\|_Y\}.$$

Also the sum $X + Y := \{x + y : \ x \in X,\ \ y \in Y\}$ is a linear subspace of $\mathcal{V}$. It is endowed with the norm

$$\|v\|_{X+Y} := \inf_{x\in X,\, y\in Y,\, x+y=v} \|x\|_X + \|y\|_Y.$$

As easily seen, $X + Y$ is isometric to the quotient space $(X \times Y)/D$, where $D = \{(x, -x) : \ x \in X \cap Y\}$. Since $\mathcal{V}$ is a Hausdorff space, then D is closed, and $X + Y$ is a Banach space. Moreover, $\|x\|_X \geq \|x\|_{X+Y}$ and $\|y\|_Y \geq \|y\|_{X+Y}$ for all $x \in X$, $y \in Y$, so that both X and Y are continuously embedded in $X + Y$.

The space $\mathcal{V}$ is used only to guarantee that $X + Y$ is a Banach space. It will disappear from the general theory.

If (X,Y) is an interpolation couple, an *intermediate space* is any Banach space E such that

$$X \cap Y \subset E \subset X + Y.$$

An *interpolation space* between X and Y is any intermediate space such that for every $T \in \mathcal{L}(X) \cap \mathcal{L}(Y)$ (that is, for every linear operator T :

$X + Y \mapsto X + Y$ whose restriction to X belongs to $\mathcal{L}(X)$ and whose restriction to Y belongs to $\mathcal{L}(Y)$), the restriction of T to E belongs to $\mathcal{L}(E)$. This implies that there is a constant independent of T such that $\|T\|_{\mathcal{L}(E)} \leq C \max\{\|T\|_{\mathcal{L}(X)}, \|T\|_{\mathcal{L}(Y)}\}$, as next lemma (taken from [63]) shows.

Lemma 0.1. *Let (X, Y) be an interpolation couple, and let E be an interpolation space between X and Y. Then there is $C > 0$ such that for each linear operator $T : X + Y \mapsto X + Y$ such that $T_{|X} \in \mathcal{L}(X)$, $T_{|Y} \in \mathcal{L}(Y)$ we have*

$$\|T\|_{\mathcal{L}(E)} \leq C \max\{\|T\|_{\mathcal{L}(X)}, \|T\|_{L(Y)}\}.$$

Proof. The space $\mathcal{B}$ of the linear operators $T : X + Y \mapsto X + Y$ such that $T_{|X} \in \mathcal{L}(X)$, $T_{|Y} \in \mathcal{L}(Y)$ is easily seen to be a Banach space with the norm $\|T\| := \max\{\|T\|_{\mathcal{L}(X)}, \|T\|_{\mathcal{L}(Y)}\}$. Indeed, if $\{T_n\}$ is a Cauchy sequence, $T_{n|X}$ converges to some T_X in $\mathcal{L}(X)$, $T_{n|Y}$ converges to some T_Y in $\mathcal{L}(Y)$, the limit operator $T : X+Y \mapsto X+Y$, $T(x+y) = T_{|X}x + T_{|Y}y$ is well defined (and obviously linear) in $X + Y$, and $T_n \to T$ in $\mathcal{B}$.

Define a linear operator $\Phi : \mathcal{B} \mapsto \mathcal{L}(E)$ by $\Phi(T) := T_{|E}$. We shall prove that the graph of Φ is closed, so that Φ is bounded and the statement follows.

Let $T_n \to T$ in $\mathcal{B}$ be such that $\Phi(T_n) \to S$ in $\mathcal{L}(E)$ as $n \to \infty$. Then for each $w \in X + Y$ and for each decomposition $w = x + y$, with $x \in X$ and $y \in Y$, we have

$$\|T_n w - T w\|_{X+Y} \leq \|(T_n - T)x\|_X + \|(T_n - T)y\|_Y \leq \|T_n - T\|(\|x\|_X + \|y\|_Y)$$

and taking the infimum over all the decompositions of w we get

$$\|T_n w - T w\|_{X+Y} \leq \|T_n - T\| \, \|w\|_{X+Y}$$

so that $\lim_{n\to\infty} T_n w = T w$ in $X + Y$. On the other hand, if $w \in E$ then $\lim_{n\to\infty} T_n w = S w$ in E and hence in $X + Y$. Therefore, $T w = S w$ for each $w \in E$, *i.e.* $S = \Phi(T)$. □

The general interpolation theory is not devoted to characterize all the interpolation spaces between X and Y but rather to construct suitable families of interpolation spaces and to study their properties. The most known and useful families of interpolation spaces are the *real interpolation spaces* which will be treated in Chapter 1, and the *complex interpolation spaces* which will be treated in Chapter 2.

Interpolation theory has a wide range of applications. We shall emphasize applications to partial differential operators and partial differential equations, referring to [90, 16] for applications to other fields. In particular we shall give self-contained proofs of optimal regularity results in Hölder and in fractional Sobolev spaces for linear elliptic and parabolic differential equations.

The domains of powers of positive operators in Banach spaces are not interpolation spaces in general, however in some interesting cases they coincide with suitable complex interpolation spaces. In any case the theory of powers of positive operators is very close to interpolation theory, and there are important connections between them. Therefore in Chapter 5 we give an elementary treatment of the powers of positive operators, with particular attention to the imaginary powers.