Introduction

These notes reproduce a series of lectures given at the Scuola Normale Superiore (Pisa), in the academic year 2017-2018, addressed to graduate students.

The extraordinary success of the Standard Model motivates the interest in deepening the structural mechanisms which are at the basis of the theory, first and foremost the *spontaneous symmetry breaking*.

The standard presentation is functional to the effectiveness of the Standard Model in producing experimentally testable predictions, via the perturbative expansion. The ensuing extraordinary agreement is by itself the best support of the theory.

Nevertheless, the still open problem of the mathematical consistency of perturbative quantum field theory, the possibility of non-perturbative effects and in any case a possible deeper grasp of the theory welcome non-perturbative information.

Furthermore, the usual discussion of the mechanism of spontaneous symmetry breaking and its application to the standard model is not free of conceptual and mathematical problems.

The aim of these notes is twofold. First to critically review the standard presentation pointing out the weak and questionable issues; as we shall see, some points of view adopted in the literature are misleading if not wrong and need corrections (at least from a foundational point of view).

Secondly, these notes offer a *non-perturbative analysis of the Higgs mechanism and of the* $U(1)$ *problem* (in the latter case without ever relying on the semiclassical instanton approximation); the aim is to provide a foundationally and mathematically sound treatment of symmetry breaking in the Standard Model, which, hopefully, corrects and/or improves the standard derivation of the final picture.

Chapter 1 is devoted to clarifying the meaning and the characterization of *spontaneous symmetry breaking*, emphasizing the crucial role of the *infinite extension of the system*, the *local structure* and the need of considering *pure phases*.

The widespread identification of spontaneous symmetry breaking merely by the non-invariance of the ground state (without further requirements) is not correct and the popular recourse to finite dimensional (mechanical) models (like e.g. a particle in a double well or in a mexican hat potential) for illustrating the phenomenon is, in our opinion, highly misleading, to say the least.

Another point of dissatisfaction is that for the breaking of a continuous group of symmetries and the related *Goldstone theorem*, the standard presentation does not sufficiently emphasize and discuss the crucial role of the *local generation of the symmetry*, at the infinitesimal level, by a conserved current.

The realization of the relevance of such a condition was the basic breakthrough which allowed Goldstone, Salam and Weinberg to transform the original semiclassical perturbative argument by Goldstone into an exact result in relativistic local quantum field theory.

Such a local generation is clearly implied by the relativistic locality of the order parameter and the symmetry generating current, but it may hold in much more general cases (including non-relativistic systems) and it qualifies as the essential ingredient for the existence of the Goldstone modes (with energy going to zero when the momentum $\mathbf{k} \to 0$).

Even if the control of such a condition at equal times is easily obtained by using the canonical commutation relations, its validity at unequal times, needed for the proof of the theorem, likely fails when *Coulomb interactions* induce a *delocalization* of the unequal time commutator of the generating current and the order parameter.

This explains the *absence of Goldstone modes* associated to symmetry breaking *in Coulomb systems*, as in superconductivity, in the plasmon jellium model and in the Higgs model in the Coulomb gauge. In our opinion, this looks a more physical explanation than the antropomorphous picture of the vector boson getting massive by eating the (would be) Goldstone boson (which, incidentally makes implicit reference to a gauge with unphysical fields).

Chapter 2 starts by discussing the puzzling issue of the possible physical meaning of breaking a gauge symmetry.

Since, by definition, a gauge symmetry reduces to the identity on the observables, it has been claimed that it cannot have a physical meaning, namely that it does not have any empirical content.

As discussed in Section 2.1, the point is that the physical description of a system involves *both* its observables *and* its states and an *unbroken global gauge symmetry* has a physical/empirical meaning displayed by a detectable characterization of the states (giving rise to superselected quantum numbers and parastatistics of the particle states).

In the case of a *broken global gauge symmetry*, a detectable physical consequence is the lack of convergence of those local observables (defined in terms of the local generators of the group), which in the unbroken case define the superselected (global) charges.

More intriguing is the meaning and/or role of a *local gauge symmetry*, since, as argued in Sections 2.3.1 and 3.2.1, it is proved to reduce to the identity not only on the observables but also on the states (apart, as we shall see, from its topological invariants).

As argued in Section 2.2, a local gauge symmetry, whose definition requires an auxiliary unphysical field algebra, plays only an intermediate role for the construction of the irreducible representations of the algebra of observables and the control of its time evolution.

In fact, the standard strategy is to start with a field algebra yielding a non-trivial representation of a local gauge group $\mathcal{G}$, and introduce a Lagrangian invariant under $\mathcal{G}$. However, the invariance under $\mathcal{G}$ must next be broken by the introduction of the gauge fixing, needed for a well posed dynamical problem (in terms of evolution equations for the local fields or of the functional integral approach). Indeed, even at the classical level, a $\mathcal{G}$ invariant Lagrangian is incompatible with a deterministic time evolutions of the gauge fields.

When the local gauge group is not fully broken by the gauge fixing, the residual local gauge group is proved to reduce to the identity on the physical states. Thus, a local gauge symmetry is doomed to lose any (direct) operational meaning at the end.

As argued in Section 2.3, the surviving counterpart of a local gauge symmetry is that the (Noether) currents which generate the corresponding global group obey *local Gauss laws on the physical states.*

The validity of local Gauss laws is a consequence of the second Noether Theorem for Lagrangians invariant under local gauge groups, but it does no longer hold in the presence of a gauge fixing; however, as discussed at length in Chapters 2, 3, the relevant point is that, independently of the gauge fixing, local Gauss laws hold in matrix elements of physical states, which may actually be characterized by the fulfillment of such a property.

It is worthwhile to stress that the characteristic general properties of gauge quantum field theories (with respect to standard quantum field theories) may all be traced back to the validity of Gauss laws on the physical states: i) *states carrying a global gauge charge cannot be localized* and charged fields cannot be local, ii) *charged particles are not Wigner particles, but infraparticles*, iii) unbroken *gauge charges are superselected*, iv) local Gauss laws lead to the *evasion of Goldstone theorem* and allow for the failure of the cluster property necessary for a linearly raising $q\,\bar{q}$ "potential".

The main issue of Chapter 2 is the Brout-Englert-Higgs mechanism, briefly the *Higgs mechanism.* The standard account of the evasion of the Goldstone theorem, based on the quadratic mean field expansion of the local gauge invariant Lagrangian, is in conflict with the non-perturbative Elitzur theorem, (a bewilderment for the students!); the role of the gauge fixing, omitted in such a simple minded explanation, is actually crucial for the setting of the renormalized perturbative series.

Quite generally, the lesson emerging from the result of Elitzur theorem makes desirable a *non-perturbative analysis of the Higgs mechanism* and results in this direction are presented in Sections 2.7, 2.8; in particular, the *absence of (physical) Goldstone bosons* is generally derived from the validity of the local Gauss law on the physical states.

A general non-perturbative result is proved in the abelian case, stating that: 1) the *global gauge group* $U(1)$ *is unbroken if and only if there are massless vector bosons* (Theor. 2.8.3), 2) in the $U(1)$ *broken case*: i) the *vector bosons are massive*, ii) the Goldstone spectrum, encoded in the two point function of the current and the order parameter, coincides with the vector boson energy-momentum spectrum, which cannot have a massless contribution, (strict *link between massive Goldstone spectrum and massive vector bosons*), iii) the flux of the field strength F_{0i} vanishes on the charged states, i.e. one has *charge screening.*

In conclusion, in this way one obtains a non-perturbative derivation of the main features derived by the perturbative approach, which takes care of the problematic points, includes the breaking by a composite field and it is foundationally and mathematically better sound.

The subject of **Chapter 3** is the $U(1)$ *problem in Quantum Chromodynamics* (QCD).

Attention is called to the problematic points of the standard solution. In particular, the common wisdom relies on the semi classical approximation of the functional integral and the topological classification of the (continuous) finite action configurations (instantons), under the problematic assumption that they dominate the functional integral (even if it is known that they have zero functional measure).

An alternative strategy is discussed which exploits *the topology of the local gauge group* $\mathcal{G}$, rather than the topological classification of the instanton solutions, improving and partly amending a previous farsighted proposal by Jackiw.

The analysis is done in the (positive definite) temporal gauge, carefully taking into account its mathematical structure, which far from being a mere technical subtlety is shown to play a crucial role for the derivation of the relevant physical properties.

The first widespread claim to be disproved is that, as a consequence of the axial anomaly, the chiral symmetry of the classical Lagrangian does not survives quantization, namely it cannot be defined as a group of transformations of the observables, which commute with time evolution.

According to G. t'Hooft, chiral symmetry is a "fictitious" or a "phoney symmetry" which exists only by using an "artificial huge" Hilbert space, which contains all the chirally transformed states. On the contrary, in Section 3.3 it is proved that it may be defined in any irreducible representation of the field algebra of the temporal gauge, which yields a *single* θ vacuum representation of the observable algebra.

In particular, the *chiral transformations are well defined on the observables and commute with time evolution*, so that one faces a genuine problem of spontaneous symmetry breaking to be confronted with the Goldstone theorem.

For the solution of the $U(1)$ problem, the chiral anomaly (present also in the abelian case) is not enough for evading the Goldstone theo-